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Abstract: The notion of τ-confluence provides a form of partial order reduction of Labelled Transition Systems (LTSs), by allowing to identify the τ-transitions whose execution does not alter the observable behaviour of the system. Several forms of τ-confluence adequate with branching bisimulation were studied in the literature, ranging from strong to weak ones according to the length of τ-transition sequences considered. Weak τ-confluence is more complex to compute than strong τ-confluence, but provides better LTS reductions. In this report, we aim at devising an efficient detection of weak τ-confluent transitions during an on-the-fly exploration of LTSs. To this purpose, we define and prove new encodings of several weak τ-confluence variants using alternation-free boolean equation systems (BESs), and we apply efficient local BES resolution algorithms to perform the detection. The resulting reduction module, developed within the CADP toolbox using the generic OPEN/CÆSAR environment for LTS exploration, was experimented on numerous examples of large LTSs underlying communication protocols and distributed systems. These experiments assessed the efficiency of the reduction and allowed us to identify the best variants of weak τ-confluence that are useful in practice.

Key-words: boolean equation system, branching bisimulation, labeled transition system, partial order reduction, on-the-fly verification

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Calcul efficace à la volée de la tau-confluence faible

Résumé : La notion de τ-confluence fournit une forme de réduction des systèmes de transitions étiquetées (STEs) par ordre partiel, en permettant d'identifier les τ-transitions dont l'exécution ne perturbe pas le comportement observable du système. Plusieurs variantes de τ-confluence adéquates avec la bisimulation de branchement ont été étudiées dans la littérature, depuis la variante forte jusqu'au variantes faibles obtenues selon la longueur des séquences de τ-transitions considérées. La τ-confluence faible est plus complexe à calculer que la τ-confluence forte, mais fournit des réductions plus fortes des STEs. Dans ce rapport, nous visons une détection efficace des τ-transitions faiblement confluentes pendant une exploration à la volée des STEs. Dans ce but, nous définissons et prouvons de nouveaux encodages de plusieurs variantes de la τ-confluence faible sous forme de systèmes d'équations booléennes (SEBs) sans alternance et nous effectuons la détection au moyen d'algorithmes efficaces de résolution locale des SEBs. Le module reducteur ainsi obtenu, developpé au sein de la boîte à outils CADP en utilisant l'environnement générique OPEN/CÆSAR pour l'exploration des STEs, a été expérimenté sur de nombreux exemples de STEs de grande taille provenant de protocoles de communication et de systèmes distribués. Ces expériences ont demontré l'efficacité de la réduction et nous ont permis d'identifier les meilleures variantes de la τ-confluence faible utiles en pratique.

Mots-clés : système d'équations booléennes, bisimulation de branchement, système de transitions étiquetées, réduction par ordre partiel, vérification à la volée

1 Introduction

Explicit-state verification consists in exploring the state space of a concurrent program by enumerating its states and transitions in order to determine whether it satisfies a temporal logic formula (*model checking*) or it is equivalent to an automaton (*equivalence checking*). Although this method allows a fast and economic detection of errors in real-life systems, in practice it is confronted with the well-known problem of *state explosion*, i.e., an exponential growth of the state space w.r.t. the size of the program under verification when it contains many concurrent processes and complex data structures. Several techniques were proposed to combat state explosion in explicit-state verification, among which on-the-fly verification (incremental construction of the state space during verification) [13, 6], partial order reduction (elimination of redundant sequences caused by the interleaving of independent transitions) [21, 10], and massively parallel verification (usage of the computing resources of several machines connected by a network) [5, 22].

Here we focus on combining on-the-fly verification and partial order reduction for the analysis of concurrent systems described using process algebraic languages, whose natural models are Labeled Transition Systems (LTSs). In this context, specific variants of partial order reduction, such as τ-confluence [11], were proposed for reducing LTSs whilst preserving branching bisimulation. Basically, confluent τ-transitions do not alter the observable behaviour of the system, and therefore they can be *prioritized* by ignoring their neighbour transitions, which reduces the amount of redundant interleavings. The on-the-fly detection of confluent τ-transitions in LTSs can be done by encoding the problem as the local resolution of an alternation-free Boolean Equation System (BES) [1, 15], which was successfully applied for strong τ-confluence [20]. The reductions can be further improved, at a higher computation cost, using weaker forms of τ-confluence, as those studied in [12, 26, 3], but no attempt of fully implementing them on-the-fly in the explicit-state setting has been done so far.

In this report, we propose a new encoding of weak τ-confluence by using alternation-free BESs. The idea is to consider τ-convergent LTSs (without cycles of τ-transitions), for which the computation of τ-closures can be done using maximal (instead of minimal) fixed point equations. These equations can be merged with those of the maximal fixed point BES encoding the confluence detection, yielding an alternation-free BES containing a single equation block. The on-the-fly detection of weakly confluent τ-transitions is carried out by combining the local BES resolution with a τ-compression [16] of the LTS, which collapses the strongly connected components (SCCs) of τ-transitions in order to obtain a τ-convergent LTS on which the BES encoding operates correctly. A similar scheme was succesfully applied to obtain alternation-free BES encodings of weak and branching bisimulation [19]. We study a hierarchy containing seven variants of weak τ-confluence, some of them not considered before in the literature, with the goal of achieving better reductions than strong τ-confluence (which is on top of the hierarchy) without penalizing the performance. For each variant R, we propose two BES encodings, an individual one that implements the mathematical definition of R, and a hierarchical one meant to speed up the convergence of BES resolution by trying to detect first the τ-transitions confluent modulo variants R' that are stronger (and therefore easier to compute) than R.

This resulted in sixteen different Bes encodings of weak τ-confluence variants, all of which were implemented within the Cadp toolbox [8] using the Open/Cæsar [7] generic environment for on-the-fly Lts manipulation. We carried out an extensive set of experiments by trying each Bes encoding of weak τ-confluence on Ltss corresponding to communication protocols and other real-life distributed systems. The measures included the memory and execution time for reducing each Lts, and also the amount of reduction achieved w.r.t. the Lts minimized modulo branching bisimulation. This enabled us to identify the Bes encodings that offer the best compromise between the computation cost and the amount of reduction achieved.

Related work. In [12], the relation between several notions of confluence and τ-inertness is explained. Besides strong and weak confluence, several other versions are discussed, which relax the confluence condition by allowing the branches to end up in two different states (instead of a single one), which are related modulo some equivalence relation (strong, weak, branching, or finite trace). In [11], an algorithm is presented to determine a set of strongly confluent τ-transitions for a given Lts, in order to reduce it; repeated application leads to increasingly smaller Ltss.

Another hierarchy of variants of τ-confluence is presented in [3]. Also here, strong and weak τ-confluence, the latter being called *ultra* weak, form the two extremes. The two variants in between, confluence and weak confluence, however, are not represented in our hierarchy, since these ones are not suitable for Bes encodings. As in our report, a method is described to detect confluent τ-transitions on-the-fly, here using a depth-first search through the confluent transition graph to determine representatives of states and detect terminal Sccs. An implementation of weak τ-confluence using this method is reported in [3, 4]; it is able to handle infinite-state spaces by means of symbolic reasoning, but considers sequences of at most two τ-transitions.

Report outline. Section 2 recalls the basic definitions of τ-confluence and introduces the hierarchy of weak τ-confluence variants that we consider. Section 3 defines the individual and hierarchical Bes encodings of the variants, and states their correctness. Section 4 briefly describes the implementation of the reductor module based on Bes resolution. Section 5 shows experimental figures and compares the performance of the encodings on various Ltss. Section 6 summarizes the results and gives directions of future work. Annex A contains the definitions of all Bes encodings and Annex B gives the proofs of their correctness.

2 Hierarchy of Weak Tau-Confluence Variants

The reduction by τ-confluence operates on Ltss, which are the natural models for action-based description languages, such as process algebras. An Lts is a tuple $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0 \rangle$, where $\mathcal{S}$ is the set of states, $\mathcal{A}$ is the set of actions (including the internal action τ), $\mathcal{T} \subseteq \mathcal{S} \times \mathcal{A} \times \mathcal{S}$ is the transition relation, and $s_0 \in \mathcal{S}$ is the initial state. A transition $(s_1, a, s_2) \in \mathcal{T}$

(also noted $s_1 \stackrel{a}{\rightarrow} s_2$) indicates that the system can move from state s_1 to state s_2 by executing action a. The notation $s_1 \stackrel{\overline{a}}{\rightarrow} s_2$ is equivalent to $s_1 \stackrel{a}{\rightarrow} s_2$ if $a \neq \tau$ and to $s_1 \stackrel{\tau}{\rightarrow} s_2 \vee s_1 = s_2$ otherwise. The definition of weak τ-confluence below is based on the notion introduced in [12] and revisited in [3] under the name ultra weak τ-confluence.

Definition 1 (Weak τ-confluence) *Given an* LTS $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0 \rangle$, *and* $S \subseteq \mathcal{S}^2$, *we say that* S *is* weak τ-confluent *in* $\mathcal{M}$ *if for every* $\langle s_1, s_2 \rangle \in S$ *and* $s_1 \stackrel{a}{\rightarrow} s_3$, *we have* $s_1 \stackrel{\tau}{\rightarrow} s_2$, *and for some* $n, m, p > 0$ *there exist* $s_{2,0}, \ldots, s_{2,n}, s'_{2,0}, \ldots, s'_{2,m}, s_4 \in \mathcal{S}$ *with* $s_{2,0} = s_2$, *and* $s_{3,0}, \ldots, s_{3,p} \in \mathcal{S}$ *with* $s_{3,0} = s_3$ *such that* $s_{2,i} \stackrel{\tau}{\rightarrow} s_{2,i+1}$ *and* $\langle s_{2,i}, s_{2,i+1} \rangle \in S$ *for* $0 \leq i < n$, $s_{2,n} \stackrel{\overline{a}}{\rightarrow} s'_{2,0}$, $s'_{2,j} \stackrel{\tau}{\rightarrow} s'_{2,j+1}$ *and* $\langle s'_{2,j}, s'_{2,j+1} \rangle \in S$ *for* $0 \leq j < m$, $s'_{2,m} \stackrel{\tau}{\rightarrow} s_4$ *and* $\langle s'_{2,m}, s_4 \rangle \in S$, $s_{3,k} \stackrel{\tau}{\rightarrow} s_{3,k+1}$ *and* $\langle s_{3,k}, s_{3,k+1} \rangle \in S$ *for* $0 \leq k < p$, *and* $s_{3,p} \stackrel{\tau}{\rightarrow} s_4$ *and* $\langle s_{3,p}, s_4 \rangle \in S$. *The* maximal τ-confluent set $\mathbb{T}(\mathcal{S}) \subseteq \mathcal{S}^2$ *is the union of all* τ*-confluent sets of* $\mathcal{M}$.

Weak τ-confluence is illustrated graphically by the R_8 diagram at the bottom of the hierarchy shown in Figure 1. The transitions drawn with solid lines are given, whereas the existence of the dashed ones must be proven in order to make the diagram confluent. Transitions that are τ-confluent (i.e., belong to S) are labeled by τ_c. The diagram R_1 at the top of the hierarchy corresponds to strong τ-confluence [11] and the other diagrams denote particular cases of weak τ-confluence obtained either by dropping sequences $s_i \stackrel{\tau_c^*}{\rightarrow} s_j$ or by replacing them with single-step sequences $s_i \stackrel{\overline{\tau_c}}{\rightarrow} s_j$. Arrows between diagrams indicate that the source diagram is a particular case of the target diagram. In [12, 11, 3] it was shown that both strong and weak τ-confluence are adequate w.r.t. branching bisimulation, i.e., if a transition $s_1 \stackrel{\tau}{\rightarrow} s_2$ is τ-confluent, then s_1 and s_2 are branching bisimilar [12, 3]. Therefore, all intermediate τ-confluence variants R_2-R_7 are also adequate w.r.t. branching bisimulation.

The τ-confluence reduction proposed in [11] consists of detecting confluent τ-transitions $s_1 \stackrel{\tau_c}{\rightarrow} s_2$ and prioritizing them by deleting all their neighbour transitions $s_1 \stackrel{a}{\rightarrow} s_3$, thus obtaining an LTS smaller, but still branching bisimilar to the original one. The detection of confluent τ-transitions during an on-the-fly exploration of the LTS can be performed efficiently using local BES resolution, as shown in the sequel.

3 BES Encodings of Weak Tau-Confluence

An alternation-free boolean equation system (BES) [1, 15] is a set of fixed point equations having boolean variables in their left-hand sides and boolean formulas in their right-hand sides. For our purpose, we consider maximal fixed point BESs containing *simple* equations [2], i.e., whose boolean formulas in the right-hand sides are either purely disjunctive, or purely conjunctive. We propose below new encodings of the weak τ-confluence variants R_2-R_8 using alternation-free BESs. For each weak τ-confluence variant R_i, we give an individual encoding derived from its definition, and several hierarchical encodings taking into account the variants stronger than R_i present in the hierarchy.

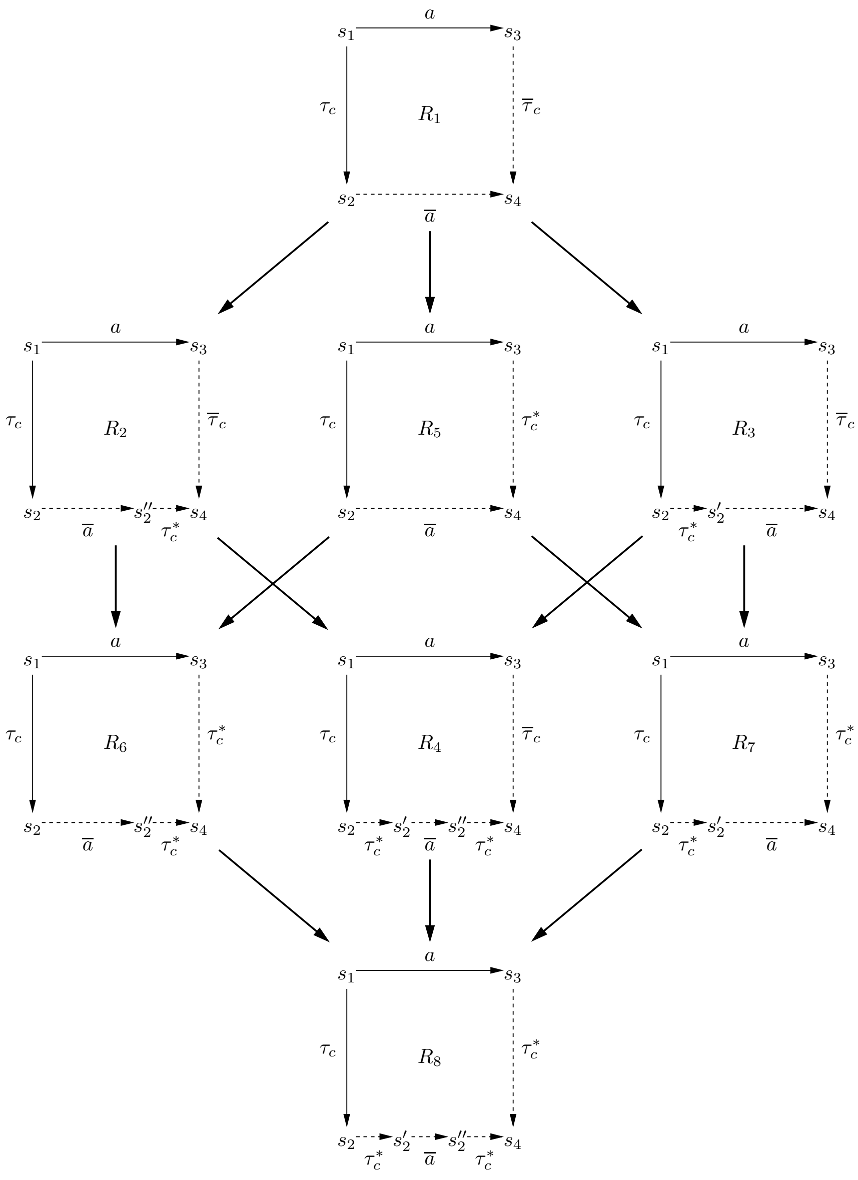

Figure 1: Variants of τ-confluence, ranging from strong (R_1) to weak (R_8)

3.1 Individual encodings

An alternation-free BES encoding for the strong τ-confluence (R_1) was proposed in [20] and applied for reducing LTSs in networks of automata. To devise a similar encoding for weak τ-confluence (R_8), we associate to each transition $s_1 \xrightarrow{\tau_c} s_2$ a boolean variable X_{s_1,s_2} indicating whether the transition is weakly τ-confluent or not. A direct encoding of the diagram R_8 yields the BES (a) below. This BES is further simplified by introducing additional variables and equations to factor subformulas, such that every right-hand side formula becomes disjunctive or conjunctive, yielding the BES (b).

$$
\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \\ & & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array} \right\}_{\substack{s_1, s_2 \in \mathcal{S} \\ a \in \mathcal{A}}}
$$

(a)

$$
\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array} \right\}_{\substack{s_1, s_2, \\ s_2', s_2'', \\ s_3, s_4 \in \mathcal{S}, \\ a \in \mathcal{A}}}
$$

(b)

We observe that the evaluation of the formulas in the right-hand sides of the equations defining $Y_{s_2,s_3,a}$, U_{s_2'',s_3}, and V_{s_3,s_4} requires to compute transitive reflexive closures over confluent τ-transitions. These τ-closure computations correspond to the evaluation of minimal fixed points and can be encoded using an additional block of minimal fixed point equations. Since these equations refer back to the variables X_{s_1,s_2} of the maximal fixed point equation block encoding the τ-confluence detection, the resulting BES would be of alternation depth two, a class with quadratic resolution complexity [24]. However, when the LTS is τ-convergent (i.e., it does not contain cycles of τ-transitions), τ-closures can be encoded using maximal (instead of minimal) fixed point equations, which can be added to the other equations of the BES (b). This produces the BES below:

$$
\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_2',s_3}) \vee \bigvee_{s_2' \xrightarrow{a} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} U'_{s_2'',s_4,s_3} \\ U'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge U_{s_4,s_3} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} V'_{s_3,s_3',s_4} \\ V'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge V_{s_3',s_4} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}
$$

The detection of confluent τ-transitions is performed by solving the X_{s_1,s_2} variables of this BES using the local resolution algorithms presented in [17]. The local BES resolution triggers a forward exploration of the LTS transitions in order to evaluate the formulas in the right-hand sides of equations, and therefore is compatible with an on-the-fly exploration of the LTS. Since this encoding scheme is correct only on τ-convergent LTSs, the cycles of τ-transitions possibly present in the LTS must be eliminated on-the-fly during the BES resolution, e.g., by using the τ-compression algorithm proposed in [16].

The BES encodings of the other weak τ-confluence variants R_2-R_7 of the hierarchy are obtained as particular cases of the BES above (see Annex A.1).

3.2 Hierarchical encodings

The BES encoding of weak τ-confluence (diagram R_8 in Figure 1) defined in Section 3.1 yields very good reductions, the resulting LTSs being sometimes very close to their minimized versions modulo branching bisimulation. However, when the LTS contains large τ-diamonds produced by the interleaved internal activity of concurrent processes, the size of the BES (number of variables and operators) may become quadratic w.r.t. the size of the LTS (number of states and transitions). On the other hand, detecting strong τ-confluence (diagram R_1) yields smaller BESs, of size linear in the number of τ-transitions and quadratic in the branching factor of the LTS [20], but always provides less reduction than the other τ-confluence variants (diagrams R_2-R_8).

To achieve the best compromise between the amount of reduction and the computational effort, one can proceed as follows: for each τ-transition encountered, first try to detect whether it is strongly confluent (R_1), then try a weaker variant of τ-confluence (e.g., R_2), then a weaker one (e.g., R_6), and finally try the weak τ-confluence itself (R_8). This reduces the average complexity of the τ-confluence detection by allowing a local BES resolution algorithm to stop as soon as it has established that the τ-transition under analysis matches some τ-confluence variant among R_1-R_8. In practice, this hierarchical detection of weak τ-confluence can be achieved by means of a BES allowing the resolution algorithms to consider in turn the desired τ-confluence variants, which amounts to follow a particular path in the diagram hierarchy shown in Figure 1.

We construct below the BES corresponding to the path R_1-R_2-R_6-R_8. Using the fact that the local BES resolution algorithms scan the right-hand sides of equations from left to right, it is sufficient to put in the right-hand sides of the equations defining X_{s_1,s_2} the boolean formulas corresponding to the four diagrams R_1, R_2, R_6, R_8 in this order:

$$\left\{ \begin{array}{lcll} X_{s_1,s_2} & \overset{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \qquad (R_1) \\ & & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \qquad (R_2) \\ & & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big) \vee \qquad (R_6) \\ & & & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big)\Big) \qquad (R_8) \end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

It is worth noticing that this BES is equivalent to (i.e., yields the same solution for the X_{s_1,s_2} variables) the BES given in Section 3.1 for the individual encoding of weak τ-confluence, because the first three disjuncts in the right-hand side formula are absorbed by (i.e., are particular cases of) the last disjunct, which corresponds to R_8. Upon simplification and factorization of common disjunctive subformulas, the BES above takes the following form (note that s_4 was renamed into s_2'' in the first disjunct):

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \overset{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \overset{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big)\Big) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array} \right\}_{\substack{s_1, s_2, \\ s_3 \in \mathcal{S}, \\ a \in \mathcal{A}}}$$

The equation defining $Y_{s_2,s_3,a}$ can be simplified by absorbing the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}$ into $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$. Such simplifications can be applied as long as they do not perturbate the evaluation order of the four cases R_1, R_2, R_6, R_8, which correspond to the disjuncts present in the right-hand side of the equation defining $Y_{s_2,s_3,a}$. Finally, after factoring τ-closures apart and encoding them using additional equations using the same scheme as in Section 3.1, we obtain the BES corresponding to the hierarchical encoding of weak τ-confluence:

$$\left\{ \begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \right) \vee U_{s_2,s_3,a} \\
Z_{s_2'',s_3} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \left(\bigvee_{s_3 \xrightarrow{\tau} s_2''} X_{s_3,s_2''} \right) \vee V_{s_2'',s_3} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & W_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} V'_{s_2'',s_4,s_3} \\
V'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge V_{s_4,s_3} \\
W_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} W'_{s_3,s_3',s_2''} \\
W'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge W_{s_3',s_2''} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} V_{s_2'',s_3} \right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\
U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a}
\end{array} \right\}_{\substack{s_1, s_2, s_2', s_2'', \\ s_3, s_3', s_4 \in \mathcal{S}, \\ a \in \mathcal{A}}}$$

We also devised the hierarchical encodings for the five other paths from R_1 to R_8, and similarly for the other variants R_2-R_7 in the hierarchy (see Annex A.2). Some of these encodings led to identical BESs, ending up with only eight different hierarchical BES encodings instead of fifteen.

4 Implementation

The BES encodings of the weak τ-confluence variants proposed in Section 3 provide the basis for an on-the-fly LTS reduction procedure. We implemented this procedure as a reductor module within the CADP[1] verification toolbox [8] using the generic OPEN/CÆSAR environment [7] for LTS exploration. This environment is centered around an *implicit* representation of LTSs defined by an API in C allowing to manipulate the transition relation as a "successor function" enumerating the transitions going out of a given state. OPEN/CÆSAR provides a rich set of primitives dedicated to graph exploration (stacks, edge lists, hash tables, etc.). It also contains the generic CÆSAR_SOLVE library [17] for local BES resolution, which operates on BESs given as *boolean graphs* [1] represented implicitly in a way similar to LTSs. This library currently offers nine resolution algorithms based on various exploration strategies (depth-first, breadth-first, etc.), each of them being able to generate diagnostics (boolean subgraphs) illustrating the truth value of a boolean variable. CÆSAR_SOLVE serves as verification engine for several on-the-fly LTS analysis tools of CADP, such as the model checker EVALUATOR 3.x [18, 17] and the equivalence checker BISIMULATOR 2.0 [19].

[1] See `http://www.inrialpes.fr/vasy/cadp`

The architecture of the reductor module is shown in Figure 4. It takes as input an implicit LTS (represented by its successor function) and produces as output the implicit LTS reduced modulo weak τ-confluence. The input LTS is first processed on-the-fly by the τ-compression module proposed in [16], which produces the implicit τ-convergent LTS obtained after collapsing the τ-SCCs and replacing them by their representative states, namely their *roots* in Tarjan's terminology [23]. This τ-convergent LTS serves as input both for an encoder module (this newly added module is about 13,300 lines of C), which translates the weak τ-confluence into an implicit BES as described in Section 3, and for an explorer module, which actually performs the detection of weakly confluent τ-transitions $s_1 \xrightarrow{\tau} s_2$ by calling the BES resolution engine CÆSAR_SOLVE to obtain the value of the boolean variables X_{s_1,s_2} of the BES. Besides prioritizing confluent τ-transitions as described in [11], the explorer module also compresses the sequences of confluent τ-transitions using the on-the-fly algorithm given in [16], which achieves additional reduction by exploiting the fact that the source and target states of each such transition are branching bisimilar. The explorer module provides as output the successor function of the reduced LTS obtained after applying these transformations.

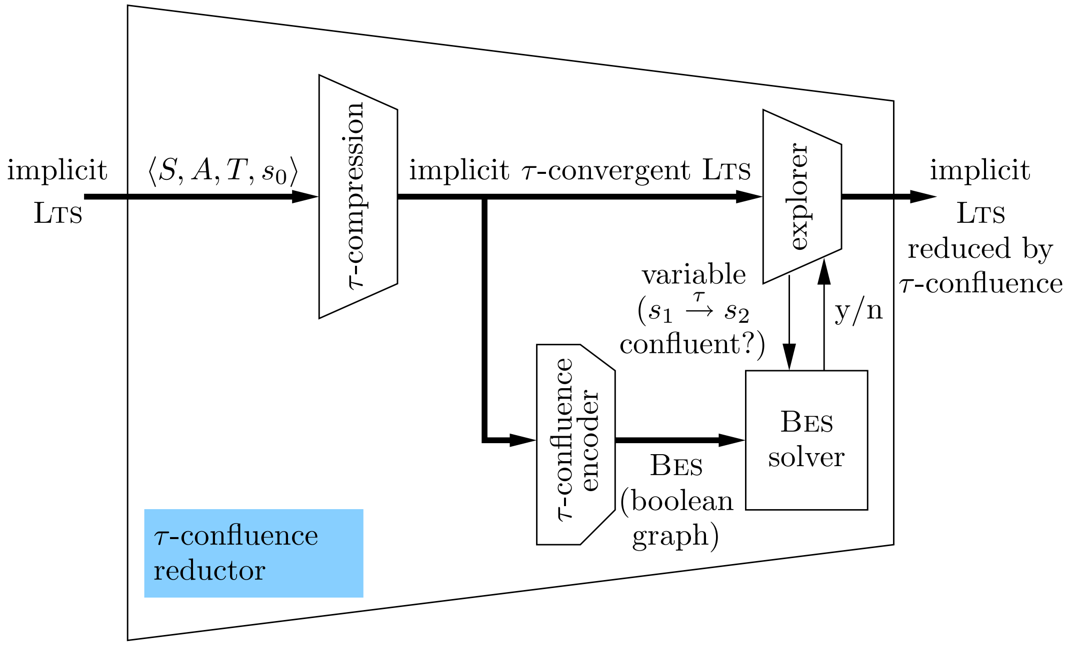

Figure 2: Architecture of the τ-confluence reductor

Currently the τ-confluence reductor module covers all the weak τ-confluence variants of the hierarchy considered in Section 2 by implementing the sixteen different BES encodings identified in Section 3 (eight individual encodings and eight hierarchical ones). Since this module acts as a filter on implicit LTSs, it can be easily plugged in the architecture of various on-the-fly verification tools of CADP in order to improve their performance by reducing the LTS simultaneously with the verification. In the next section, we show the application of this module for LTS generation only, which transforms an implicit LTS into a (reduced) explicit one, represented as a file in the BCG (*Binary Coded Graphs*) format of CADP.

5 Experimental Results

We studied the performance of all the individual and hierarchical BES encodings extensively. For this, we used around 30 LTSs from the VLTS benchmark suite[2] stemming from industrial case studies and 7 communication protocols taken from the CADP demo examples[3]. All experiments were run on a LINUX machine with a 2.2 GHz CPU and 16 Gb of memory. Here, we present in several graphs the most interesting results obtained, concerning the individual encodings of R_1 and R_8, and four hierarchical encodings, namely $R_{1\text{-}2\text{-}4}$, $R_{1\text{-}3\text{-}7}$, $R_{1\text{-}5\text{-}7\text{-}8}$, and $R_{1\text{-}3\text{-}4\text{-}8}$.

In the graph at the top of Figure 5, the different sizes of the output LTSs are compared to the usual sizes without any confluence checking ("full") and the minimal sizes obtainable by reducing the full LTSs modulo branching bisimulation ("min"). We observe that for many of the VLTS cases displayed, there are no big differences between the sizes, while for the case studies there are. Most of the VLTS examples do not stem from communication protocols, and they seem less sensitive to which encoding we use.

The graphs at the middle and the bottom of Figure 5 show the execution times of the different encodings when using two different resolution algorithms available in the CÆSAR_SOLVE library [17, 19] of CADP, namely A5 and A8, which exhibit the best performance for solving BESs encoding τ-confluence detection. Clearly, besides R_1, which produces the worst reduction, encodings $R_{1\text{-}2\text{-}4}$ and $R_{1\text{-}3\text{-}7}$ are on average the fastest. The experiments point out that incorporating R_8 into a hierarchical encoding can be particularly costly with respect to time, hence it should only be done if the gain in reduction of the output LTS justifies it. The performance of the other two hierarchical encodings excluding R_8 tends to be in between the performance of these two.

The last two graphs shown in Figure 5 display the number of boolean variables created in memory in order to solve the BESs using algorithms A5 and A8, respectively. Algorithm A8 tends to be more economic in memory, which comes at the price of longer execution times.

Based on these results, we conclude that on average the performance of R_i, with $i \in \{4, 7, 8\}$, is worse than any hierarchical encoding incorporating R_i. Concerning R_8, encoding $R_{1\text{-}3\text{-}4\text{-}8}$ is often most efficient, but usually still very costly, and the same or slightly worse results can be obtained more efficiently using an encoding to R_7, particularly $R_{1\text{-}3\text{-}7}$. Looking at Figure 1, our experiments point out that if we want to incrementally check for confluences weaker than strong, we should first try to detect strong confluence, then try to detect a sequence of confluent τ-transitions starting at s_2, i.e., the difference between R_3 and R_1, then if desired try to detect a sequence of confluent τ-transitions after the following $\overline{a}$, as in R_4, and finally extend the check of $s_3 \xrightarrow{\overline{\tau_c}} s_4$ to $s_3 \xrightarrow{\tau_c^*} s_4$ (encoding $R_{1\text{-}3\text{-}4}$, not shown in

[2]See `http://www.inrialpes.fr/vasy/cadp/resources/benchmark_bcg.html`

[3]See `http://www.inrialpes.fr/vasy/cadp/demos/demo_XX.html`

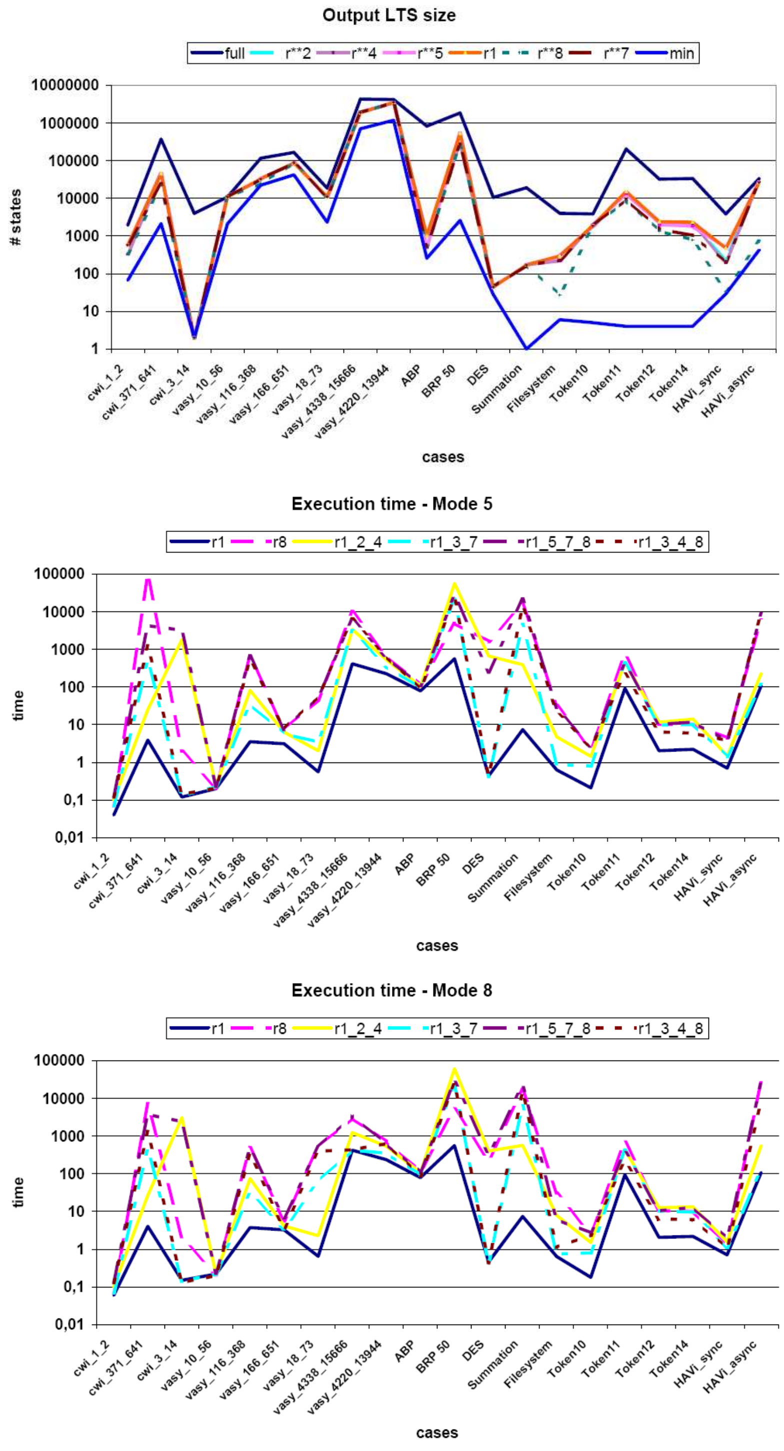

Figure 3: Performance of weak τ-confluence reductions (I)

the graphs, is usually faster than encoding $R_{1\text{-}2\text{-}4}$). Furthermore, it seems that leaving out either the check for $s_3 \xrightarrow{\tau_c^*} s_4$ or $s_2'' \xrightarrow{\tau_c^*} s_4$ speeds the resolution up considerably, the first being a better check to perform concerning the size of the output LTS. This is supported by the fact that particularly encodings R_4 and R_8 are expensive to perform, while R_5 and R_7 provide a better tradeoff between time and LTS size. Leaving $s_2'' \xrightarrow{\tau_c^*} s_4$ out, as is done in hierarchical encodings of R_7, tends to produce slightly bigger LTSs, but does so several orders of magnitude faster.

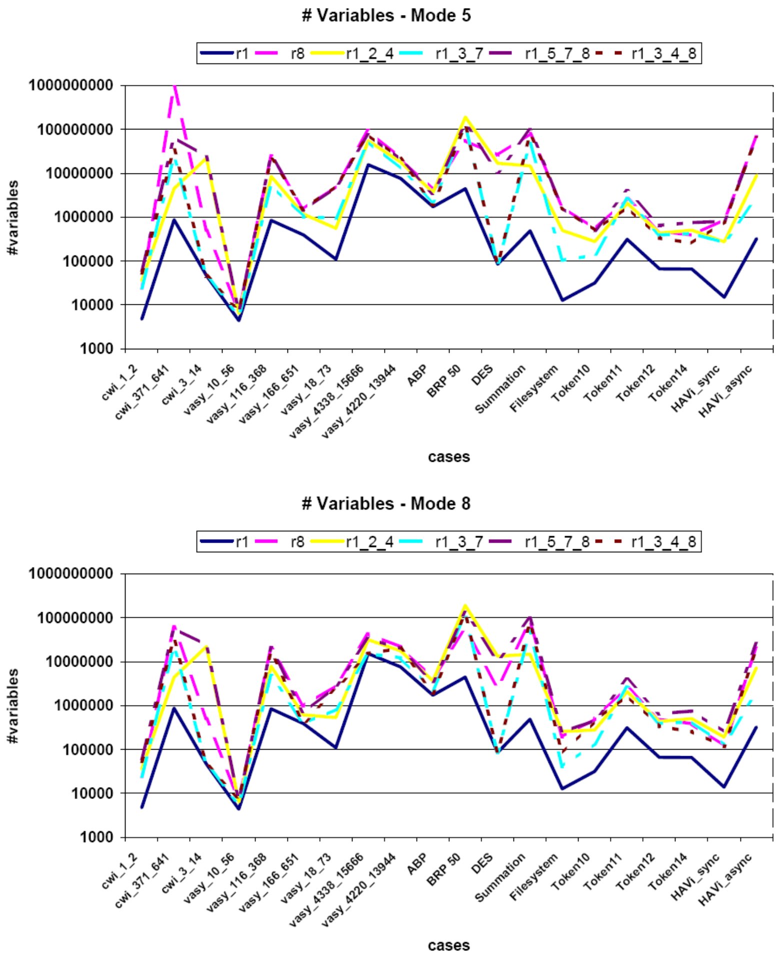

Figure 4: Performance of weak τ-confluence reductions (II)

6 Conclusion and Future Work

The possibility of efficiently reducing an LTS on-the-fly while preserving branching bisimulation is a useful feature for equivalence checkers and model checkers. In this report we studied a hierarchy of weak τ-confluence variants and proposed new (individual and hierarchical) encodings of these variants in terms of alternation-free BESs, which provided the basis of a reduction procedure based on local BES resolution. The resulting reductor module, built using the OPEN/CÆSAR [7] environment of CADP [8] and the CÆSAR_SOLVE [17] boolean resolution library, can be used as accelerator for various on-the-fly verification tools. The experiments we carried out on benchmark LTSs coming from real-life case-studies enabled us to identify the combinations of weak τ-confluence variants and BES encodings that provide the best compromise between the amount of reduction achieved and the complexity of its computation. This improves over existing results in this setting, which concerned mainly strong τ-confluence and (to a limited extent only) weak τ-confluence.

We plan to continue our work by first experimenting the new reductor module in conjunction with other on-the-fly verification tools of CADP, such as the (sequential) model checker EVALUATOR 3.x [18, 17] and equivalence checker BISIMULATOR 2.0 [17, 19] in order to determine the most appropriate weak τ-confluent variant suitable for these tools. Another line of work concerns the combination of the reductor module with the DISTRIBUTOR tool [9] for distributed state space generation, which currently provides only τ-compression and strong τ-confluence as reduction features. Finally, we will seek to devise BES encodings for other forms of τ-confluence, such as those defined in [12].

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A Encodings of the τ-confluence variants

A.1 Individual encodings

Encoding of $\mathbf{R_1}$ A direct encoding of the diagram R_1 yields the BES below:

$$\left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \end{array} \right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_2}$ A direct encoding of the diagram R_2 yields the BES below:

$$\left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_3,s_4} \\ Z'_{s_2'',s_3,s_4} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge Z_{s_4,s_3} \\ U_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_2''} X_{s_3,s_2''} \end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_3}$ A direct encoding of the diagram R_3 yields the BES below:

$$\left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_2',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_3,s_2'}) \vee \bigvee_{s_2' \xrightarrow{a} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \end{array} \right\}_{s_1,s_2,s_2',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_4}$ A direct encoding of the diagram R_4 yields the BES below:

$$\left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \stackrel{\tau}{\rightarrow} s_2'} Y'_{s_2,s_2',s_3,a} \\
Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\
Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_2',s_3}) \vee \bigvee_{s_2' \stackrel{a}{\rightarrow} s_2''} U_{s_2'',s_3} \\
U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \stackrel{\tau}{\rightarrow} s_4} U'_{s_2'',s_3,s_4} \\
U'_{s_2'',s_3,s_4} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge U_{s_4,s_3} \\
V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \stackrel{\tau}{\rightarrow} s_4} X_{s_3,s_4}
\end{array}\right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_5}$ A direct encoding of the diagram R_5 yields the BES below:

$$\left\{X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} \bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}\right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_4} Z_{s_3,s_4} \\
Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}
\end{array}\right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \bigvee_{s_2 \stackrel{a}{\rightarrow} s_4} Z_{s_3,s_4} \\
Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \stackrel{\tau}{\rightarrow} s_3'} Z'_{s_3,s_3',s_4} \\
Z'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge Z_{s_3',s_4}
\end{array}\right\}_{s_1,s_2,s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_6}$ A direct encoding of the diagram R_6 yields the BES below:

$$\left\{X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} \bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}\right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_3,s_4} \\ Z'_{s_2'',s_3,s_4} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge Z_{s_4,s_3} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} U'_{s_3,s_3',s_4} \\ U'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge U_{s_3',s_4} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of $\mathbf{R_7}$ A direct encoding of the diagram R_7 yields the BES below:

$$\left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \right\}_{s_1,s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

Simplifying the BES such that every equation has a single boolean operator in its right-hand side yields the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding the τ-closures using maximal fixed point variables produces the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_3,s_2'}) \vee \bigvee_{s_2' \xrightarrow{a} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} U'_{s_3,s_3',s_4} \\ U'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge U_{s_3',s_4} \end{array}\right\}_{s_1,s_2,s_2',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

A.2 Hierarchical encodings

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_2}$-$\mathbf{R_4}$-$\mathbf{R_8}$ Combining the boolean formulas corresponding to the four diagrams R_1, R_2, R_4, R_8 yields the following:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\to} s_3} \begin{array}{l} \left(\left(\bigvee_{s_2 \stackrel{\overline{a}}{\to} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true}\right) \vee \right. \\ \left(\bigvee_{s_2 \stackrel{\overline{a}}{\to} s_2'' \stackrel{\tau_c^*}{\to} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true}\right) \vee \\ \left(\bigvee_{s_2 \stackrel{\tau_c^*}{\to} s_2' \stackrel{\overline{a}}{\to} s_2'' \stackrel{\tau_c^*}{\to} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true}\right) \vee \\ \left.\left(\bigvee_{s_2 \stackrel{\tau_c^*}{\to} s_2' \stackrel{\overline{a}}{\to} s_2'' \stackrel{\tau_c^*}{\to} s_4} \bigvee_{s_3 \stackrel{\tau_c^*}{\to} s_4} \mathsf{true}\right)\right) \end{array} \end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the first disjunct):

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\to} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\overline{a}}{\to} s_2''} \left(\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \stackrel{\tau_c^*}{\to} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true}\right) \vee \\ & & \bigvee_{s_2 \stackrel{\tau_c^*}{\to} s_2'} \bigvee_{s_2' \stackrel{\overline{a}}{\to} s_2''} \bigvee_{s_2'' \stackrel{\tau_c^*}{\to} s_4} \left(\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true} \vee \bigvee_{s_3 \stackrel{\tau_c^*}{\to} s_4} \mathsf{true}\right) \end{array} \right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true}$ by $\bigvee_{s_3 \stackrel{\tau_c^*}{\to} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\to} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \stackrel{\overline{a}}{\to} s_2''} Z_{s_2'',s_3}\right) \vee U_{s_2,s_3,a} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \stackrel{\tau_c^*}{\to} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\to} s_4} \mathsf{true} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\tau_c^*}{\to} s_2'} \bigvee_{s_2' \stackrel{\overline{a}}{\to} s_2''} W_{s_2'',s_3} \\ W_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \stackrel{\tau_c^*}{\to} s_4} S_{s_3,s_4} \\ S_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\tau_c^*}{\to} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the BES below:

$$
\left\{
\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \right) \vee U_{s_2,s_3,a} \\
Z_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_4,s_3} \\
Z'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge Z_{s_4,s_3} \\
V_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_2''} X_{s_3,s_2''} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge W_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} W_{s_2'',s_3} \right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\
U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\
W_{s_2'',s_3} & \stackrel{\nu}{=} & S_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} W'_{s_2'',s_4,s_3} \\
W'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge W_{s_4,s_3} \\
S_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} S'_{s_3,s_3',s_2''} \\
S'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge S_{s_3',s_2''}
\end{array}
\right\}_{\substack{s_1, s_2, s_2', s_2'', \\ s_3, s_3', s_4 \in \mathcal{S}, \\ a \in \mathcal{A}}}
$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_5}$-$\mathbf{R_6}$-$\mathbf{R_8}$ Combining the boolean formulas corresponding to the four diagrams R_1, R_5, R_6, R_8 yields the following:

$$
\left\{
\begin{array}{rcll}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \Big) \vee \\
 & & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \vee \\
 & & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \vee \\
 & & & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \Big)
\end{array}
\right\}_{s_1,s_2 \in \mathcal{S}}
$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the first and second disjunct):

$$
\left\{
\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \vee \\
 & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}
\end{array}
\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}
$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$
\left\{
\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\
Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} U_{s_2'',s_3} \\
U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \\
V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}
\end{array}
\right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}
$$

Finally, after encoding τ-closures using boolean equations, we obtain a BES identical to the encoding of R_8:

$$\left\{ \begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\
Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\
Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_2',s_3}) \vee \bigvee_{s_2' \xrightarrow{a} s_2''} U_{s_2'',s_3} \\
U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} U'_{s_2'',s_4,s_3} \\
U'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge U_{s_4,s_3} \\
V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} V'_{s_3,s_3',s_4} \\
V'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge V_{s_3',s_4}
\end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_5}$-$\mathbf{R_7}$-$\mathbf{R_8}$ Combining the boolean formulas corresponding to the four diagrams R_1, R_5, R_7, R_8 yields the following:

$$\left\{ \begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\
& & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big) \vee \\
& & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big) \vee \\
& & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big)\Big)
\end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the first and second disjunct):

$$\left\{ \begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true}\Big) \vee \\
& & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big) \vee
\end{array} \right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{ \begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_3,s_2''}\Big) \vee U_{s_2,s_3,a} \\
Z_{s_3,s_2''} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} V_{s_2'',s_3} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & Z_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} Z_{s_3,s_4}
\end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the BES below:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_3,s_2''}\right) \vee U_{s_2,s_3,a} \\
Z_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} Z'_{s_3,s_3',s_2''} \\
Z'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge Z_{s_3',s_2''} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} V_{s_2'',s_3}\right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\
U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & Z_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} V'_{s_2'',s_4,s_3} \\
V'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge V_{s_4,s_3}
\end{array}\right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_3}$-$\mathbf{R_4}$-$\mathbf{R_8}$ Combining the boolean formulas corresponding to the four diagrams R_1, R_3, R_4, R_8 yields the following:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Bigg(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right) \Bigg)
\end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the second disjunct):

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right)\right)
\end{array}\right\}_{\substack{s_1,\, s_2, \\ s_3 \in \mathcal{S}, \\ a \in \mathcal{A}}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\
Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} V_{s_2'',s_3} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & Z_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} W_{s_3,s_4} \\
W_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}
\end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the BES below (note that on the right hand side of V_{s_2'',s_3}, the disjunct $Z_{s_3,s_2''}$ is absorbed by $W_{s_3,s_2''}$):

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\
Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} V_{s_2'',s_3}\right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\
U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & W_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} V'_{s_2'',s_4,s_3} \\
V'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge V_{s_4,s_3} \\
W_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} W'_{s_3,s_3',s_2''} \\
W'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge W_{s_3',s_2''}
\end{array}\right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_3}$-$\mathbf{R_7}$-$\mathbf{R_8}$ Combining the boolean formulas corresponding to the four diagrams R_1, R_3, R_7, R_8 yields the following:

$$\left\{\begin{array}{rcll}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \left(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \right. \\
& & & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right) \vee \\
& & & \left.\left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right)\right)
\end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the second and third disjunct):

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\
& & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right)
\end{array}\right\}_{\substack{s_1, s_2, \\ s_3 \in \mathcal{S}, \\ a \in \mathcal{A}}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl}
X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\
Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \\
U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} V_{s_2'',s_3} \\
V_{s_2'',s_3} & \stackrel{\nu}{=} & W_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} W_{s_3,s_4} \\
W_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}
\end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a Bes identical to the one for R_1-R_3-R_4-R_8:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} V_{s_2'',s_3}\right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\ U'_{s_2.s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\ V_{s_2'',s_3} & \stackrel{\nu}{=} & W_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} V'_{s_2'',s_4,s_3} \\ V'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge V_{s_4,s_3} \\ W_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} W'_{s_3,s_3',s_2''} \\ W'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge W_{s_3',s_2''} \end{array}\right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_2}$ Combining the boolean formulas corresponding to the two diagrams R_1, R_2 yields the following:

$$\left\{\begin{array}{rcll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \left(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \right. \\ & & & \left.\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right)\right) \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the Bes above becomes as follows (note that s_4 was renamed into s_2'' in the first disjunct):

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (factoring τ-closures apart into equations) yield the following Bes:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} U_{s_3,s_4} \\ U_{s_3,s_2''} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a Bes identical to the one for R_2:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3}\right) \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_4,s_3} \\ Z'_{s_2''.s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \vee Z_{s_4,s_3} \\ U_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_2''} X_{s_3,s_2''} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_5}$ Combining the boolean formulas corresponding to the two diagrams R_1, R_5 yields the following:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big)\Big) \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\Big) \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a BES identical to the one for R_5:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} Z'_{s_3,s_3',s_4} \\ Z'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge Z_{s_3',s_4} \end{array}\right\}_{s_1,s_2,s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_3}$ Combining the boolean formulas corresponding to the two diagrams R_1, R_3 yields the following:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big)\Big) \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows:

$$\left\{\begin{array}{lll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \stackrel{\tau_c^*}{\rightarrow} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \stackrel{\overline{a}}{\rightarrow} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a BES identical to the encoding of R_3:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \stackrel{\tau}{\rightarrow} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_3,s_2'}) \vee \bigvee_{s_2' \stackrel{a}{\rightarrow} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \stackrel{\tau}{\rightarrow} s_4} X_{s_3,s_4} \end{array}\right\}_{s_1,s_2,s_2',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_2}$-$\mathbf{R_6}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_2, R_6 yields the following:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} \begin{array}{l} \Big(\Big(\bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_4} \mathsf{true}\Big) \vee \\ \Big(\bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_4} \mathsf{true}\Big) \vee \\ \Big(\bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}\Big)\Big) \end{array} \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the first disjunct):

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_2''} \Big(\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} \Big(\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_4} \mathsf{true} \vee \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}\Big)\Big) \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_4} \mathsf{true}$ by $\bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \stackrel{a}{\rightarrow} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\overline{a}}{\rightarrow} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true} \\ V_{s_3,s_2''} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\overline{\tau_c}}{\rightarrow} s_2''} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a BES identical to the encoding of R_6 (note that on the right hand side of Z_{s_2'',s_3}, the disjunct $V_{s_3,s_2''}$ is absorbed by $U_{s_3,s_2''}$):

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_4,s_3} \\ Z'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge Z_{s_4,s_3} \\ U_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} U'_{s_3,s_3',s_2''} \\ U'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge U_{s_3',s_2''} \end{array}\right\}_{s_1,s_2,s_2'',s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_5}$-$\mathbf{R_6}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_5, R_6 yields the following:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Bigg(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \Bigg) \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows (note that s_4 was renamed into s_2'' in the first and second disjunct):

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \Big) \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_2''} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} U_{s_3,s_4} \\ U_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain a BES identical to the encoding of R_6:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & U_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} Z'_{s_2'',s_4,s_3} \\ Z'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge Z_{s_4,s_3} \\ U_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} U'_{s_3,s_3',s_2''} \\ U'_{s_3,s_3',s_2''} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge U_{s_3',s_2''} \end{array}\right\}_{s_1,s_2,s_2'',s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_2}$-$\mathbf{R_4}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_2, R_4 yields the following:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \Big(\Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \Big(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big)\Big) \end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the Bes above becomes as follows (note that s_4 was renamed into s_2'' in the first disjunct):

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} \Big(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\Big) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (factoring τ-closures apart into equations) yield the following Bes:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \Big(\bigvee_{s_2 \xrightarrow{\overline{a}} s_2''} Z_{s_2'',s_3}\Big) \vee U_{s_2,s_3,a} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee W_{s_2'',s_3} \\ V_{s_3,s_2''} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} W_{s_2'',s_3} \\ W_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the Bes below:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_2,s_3}) \vee \Big(\bigvee_{s_2 \xrightarrow{a} s_2''} Z_{s_2'',s_3}\Big) \vee U_{s_2,s_3,a} \\ Z_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee W_{s_2'',s_3} \\ V_{s_3,s_2''} & \stackrel{\nu}{=} & (s_3 = s_2'') \vee \bigvee_{s_3 \xrightarrow{\tau} s_2''} X_{s_3,s_2''} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge W_{s_2,s_3}) \vee \Big(\bigvee_{s_2 \xrightarrow{a} s_2''} W_{s_2'',s_3}\Big) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\ U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\ W_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} W'_{s_2'',s_4,s_3} \\ W'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge W_{s_4,s_3} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_3}$-$\mathbf{R_4}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_3, R_4 yields the following:

$$\left\{ \begin{array}{lcll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \left(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \vee \right. \\ & & & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \vee \\ & & & \left. \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \right) \end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the Bes above becomes as follows (note that s_4 was renamed into s_2'' in the second disjunct):

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_2''} \mathsf{true} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \end{array} \right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (factoring τ-closures apart into equations) yield the following Bes:

$$\left\{ \begin{array}{lcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4} \right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} V_{s_2'',s_3} \\ V_{s_2'',s_3} & \stackrel{\nu}{=} & Z_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} Z_{s_3,s_4} \end{array} \right\}_{s_1,s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the Bes below:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4} \right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_2,s_3}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_2''} V_{s_2'',s_3} \right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\ U'_{s_2.s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\ V_{s_2'',s_3} & \stackrel{\nu}{=} & Z_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} V'_{s_2'',s_4,s_3} \\ V'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge V_{s_4,s_3} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_3}$-$\mathbf{R_7}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_3, R_7 yields the following:

$$\left\{ \begin{array}{lcll} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} & \left(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \vee \right. \\ & & & \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \right) \vee \\ & & & \left. \left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \right) \right) \end{array} \right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right) \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following BES:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the BES below:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4}\right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_4} X_{s_3,s_4} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge V_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} V_{s_3,s_4}\right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\ U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} V'_{s_3,s_3',s_4} \\ V'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge V_{s_3',s_4} \end{array}\right\}_{s_1,s_2,s_2',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Encoding of the path $\mathbf{R_1}$-$\mathbf{R_5}$-$\mathbf{R_7}$ Combining the boolean formulas corresponding to the three diagrams R_1, R_5, R_7 yields the following:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} \left(\left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}\right) \vee \right. \\ & & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right) \vee \\ & & \left.\left(\bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right)\right) \end{array}\right\}_{s_1,s_2 \in \mathcal{S}}$$

Upon simplification and factorization of common disjunctive subformulas, the BES above becomes as follows:

$$\left\{\begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\overline{a}} s_4} \left(\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true} \vee \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\right) \vee \\ & & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array}\right\}_{s_1,s_2,s_3 \in \mathcal{S}, a \in \mathcal{A}}$$

Further simplifications (absorption of the disjunct $\bigvee_{s_3 \xrightarrow{\overline{\tau_c}} s_4} \mathsf{true}$ by $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$ and factoring τ-closures apart into equations) yield the following Bes:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \left(\bigvee_{s_2 \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4} \right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} \bigvee_{s_2' \xrightarrow{\overline{a}} s_4} Z_{s_3,s_4} \end{array} \right\}_{s_1,s_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Finally, after encoding τ-closures using boolean equations, we obtain the Bes below:

$$\left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4} \right) \vee U_{s_2,s_3,a} \\ Z_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} Z'_{s_3,s_3',s_4} \\ Z'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge Z_{s_3',s_4} \\ U_{s_2,s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge Z_{s_3,s_2}) \vee \left(\bigvee_{s_2 \xrightarrow{a} s_4} Z_{s_3,s_4} \right) \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} U'_{s_2,s_2',s_3,a} \\ U'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge U_{s_2',s_3,a} \end{array} \right\}_{s_1,s_2,s_2',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

B Proofs of the BES Encodings

We prove in the sequel the correctness of our BES encoding for weak τ-confluence (R_8 in Figure 1). The correctness proofs for encodings of τ-confluences $R_1, \ldots, R_7$ and the hierarchical encodings are very similar to the one for R_8 and are therefore left as exercise for the interested reader.

A few additional notions are needed in order to proceed. Let $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0 \rangle$ be an LTS. Consider the two lattices $\langle \mathsf{Bool}^{|\mathcal{S}|^2}, \sqsubseteq, \mathsf{false}^{|\mathcal{S}|^2}, \mathsf{true}^{|\mathcal{S}|^2}, \sqcup, \sqcap \rangle$ and $\langle 2^{\mathcal{S}^2}, \subseteq, \emptyset, \mathcal{S}^2, \cup, \cap \rangle$, where the relation $\sqsubseteq$ and the operations $\sqcup$, $\sqcap$ are defined as the pointwise extensions of the boolean connectors $\Rightarrow$, $\vee$, and $\wedge$, respectively. These lattices are isomorphic, being related by the function $\Gamma : \mathsf{Bool}^{|\mathcal{S}|^2} \to 2^{\mathcal{S}^2}$ defined below:

$$\Gamma(\langle b_{s_1,s_2} \rangle_{s_1,s_2 \in \mathcal{S}}) = \{ \langle s_1, s_2 \rangle \mid b_{s_1,s_2} = \mathsf{true} \}.$$

It is straightforward to show that Γ is an isomorphism, i.e., it is a bijection preserving the compatibility of operations ($\boldsymbol{b} \sqsubseteq \boldsymbol{b}' \Leftrightarrow \Gamma(\boldsymbol{b}) \subseteq \Gamma(\boldsymbol{b}')$, $\Gamma(\mathsf{false}^{|\mathcal{S}|^2}) = \emptyset$, $\Gamma(\mathsf{true}^{|\mathcal{S}|^2}) = \mathcal{S}^2$, $\Gamma(\boldsymbol{b} \sqcup \boldsymbol{b}') = \Gamma(\boldsymbol{b}) \cup \Gamma(\boldsymbol{b}')$, and $\Gamma(\boldsymbol{b} \sqcap \boldsymbol{b}') = \Gamma(\boldsymbol{b}) \cap \Gamma(\boldsymbol{b}')$).

Furthermore, we define that $b_{s_1,s_2} = \mathsf{true}$ iff $s_1 \xrightarrow{\tau_c} s_2$, i.e. b_{s_1,s_2} expresses whether there exists a confluent τ-transition between states s_1 and s_2.

We provide below a definition of weak τ-confluence, based on the definitions of τ-confluence by [20] and ultra weak confluence by [3], and recall two corresponding BES encodings given in Section 3.1, namely the direct encoding of diagram R_8, and the encoding such that each equation contains a single boolean operator, and τ-closures are encoded using maximal fixed point variables.

Definition 2 (Weak τ-confluence) *Given an* LTS $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0 \rangle$, *and* $S \subseteq \mathcal{S}^2$, *we say that S is* weak τ-confluent *in $\mathcal{M}$ if for every $\langle s_1, s_2 \rangle \in S$ and $s_1 \xrightarrow{a} s_3$, we have $s_1 \xrightarrow{\tau} s_2$, and for some $n, m, p > 0$ there exist $s_{2,0}, \ldots, s_{2,n}, s'_{2,0}, \ldots, s'_{2,m}, s_4 \in \mathcal{S}$ with $s_{2,0} = s_2$, and $s_{3,0}, \ldots, s_{3,p} \in \mathcal{S}$ with $s_{3,0} = s_3$ such that $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ and $\langle s_{2,i}, s_{2,i+1} \rangle \in S$ for $0 \leq i < n$, $s_{2,n} \xrightarrow{\overline{a}} s'_{2,0}$, $s'_{2,j} \xrightarrow{\tau} s'_{2,j+1}$ and $\langle s'_{2,j}, s'_{2,j+1} \rangle \in S$ for $0 \leq j < m$, $s'_{2,m} \xrightarrow{\tau} s_4$ and $\langle s'_{2,m}, s_4 \rangle \in S$, $s_{3,k} \xrightarrow{\tau} s_{3,k+1}$ and $\langle s_{3,k}, s_{3,k+1} \rangle \in S$ for $0 \leq k < p$, and $s_{3,p} \xrightarrow{\tau} s_4$ and $\langle s_{3,p}, s_4 \rangle \in S$.*

The maximal τ-confluent set $\mathbb{T}(\mathcal{S}) \subseteq \mathcal{S}^2$ *is the union of all τ-confluent sets of $\mathcal{M}$.*

Definition 3 (Directly encoded weak τ-confluence BES) *Let $(\mathcal{S}, \mathcal{A}, \mathcal{T}, s_0)$ be a τ-convergent* LTS. *The weak τ-confluent set $\mathbb{T}(\mathcal{S}) \subseteq \mathcal{S}^2$ is directly encoded by the maximal fixed point* BES *below:*

$$B_{wtc1} = \left\{ X_{s_1,s_2} \stackrel{\nu}{=} \bigwedge_{s_1 \xrightarrow{a} s_3} \; \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2 \xrightarrow{\overline{a}} s''_2 \xrightarrow{\tau_c^*} s_4} \; \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \right\}_{s_1, s_2 \in \mathcal{S}, a \in \mathcal{A}}$$

The interpretation $[\![B_{wtc1}]\!]$ *is defined as the maximal fixed point* $\nu\Phi_{wtc1}$*, where* Φ_{wtc1} : $\mathsf{Bool}^{|\mathcal{S}|^2} \to \mathsf{Bool}^{|\mathcal{S}|^2}$ *is the (monotonic) functional associated to* B_{wtc1}*:*

$$\Phi_{wtc1}(\langle b_{s_1,s_2}\rangle_{s_1,s_2\in\mathcal{S}}) \quad = \quad \langle [\![\bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2' \xrightarrow{\overline{a}} s_2'' \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}]\!][b_{s_1,s_2}/X_{s_1,s_2}]\rangle_{s_1,s_2\in\mathcal{S}}.$$

Definition 4 (Weak τ-confluence BES) *Let* $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0\rangle$ *be a* τ*-convergent* LTS*. The weak* τ*-confluent set* $\mathbb{T}(\mathcal{S}) \subseteq \mathcal{S}^2$ *is encoded by the maximal fixed point* BES *below:*

$$B_{wtc} = \left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_2',s_3}) \vee \bigvee_{s_2' \xrightarrow{a} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} U'_{s_2'',s_4,s_3} \\ U'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge U_{s_4,s_3} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} V'_{s_3,s_3',s_4} \\ V'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge V_{s_3',s_4} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4\in\mathcal{S}, a\in\mathcal{A}}$$

We first define some notations, and then we prove the correctness of the BES encodings for weak τ-confluence.

Definition 5 *Let* $\mathcal{X}$ *be a set of boolean variables including* $X_1, ..., X_n, X_{n+1}$*. Let* $B = \{X_i \stackrel{\nu}{=} \varphi_i\}_{1\le i\le n}$ *and* $B' = \{X_i \stackrel{\nu}{=} \varphi_i\}_{1\le i\le n+1}$ *be two* BES*s having their first* n *equations identical (*φ_i *are boolean formulas built from disjunctions and conjunctions) and* $\delta : \mathcal{X} \to \mathsf{Bool}$ *be a context. Let* $\Phi_\delta : \mathsf{Bool}^n \to \mathsf{Bool}^n$ *and* $\Phi'_\delta : \mathsf{Bool}^{n+1} \to \mathsf{Bool}^{n+1}$ *be the two functionals associated to* B *and* B' *in the context* δ*:*

$$\Phi_\delta(\langle b_i\rangle_{1\le i\le n}) = \langle [\![\varphi_i]\!](\delta \oslash [b_j/X_j]_{1\le j\le n})\rangle_{1\le i\le n}$$
$$\Phi'_\delta(\langle b_i\rangle_{1\le i\le n+1}) = \langle [\![\varphi_i]\!](\delta \oslash [b_j/X_j]_{1\le j\le n+1})\rangle_{1\le i\le n+1}$$

where $\delta \oslash [b_j/X_j]_{1\le j\le n}$ *denotes a context identical to* δ *except for variables* $X_1, ..., X_n$*, which are assigned values* $b_1, ..., b_n$*. According to Kleene's theorem [14], the maximal fixed points of the functionals* Φ_δ *and* Φ'_δ *can be computed as follows:*

$$\nu\Phi_\delta = \sqcap_{k\ge 0}\Phi_\delta^k(\mathsf{true}^n) \qquad \nu\Phi'_\delta = \sqcap_{k\ge 0}{\Phi'_\delta}^k(\mathsf{true}^{n+1}).$$

The notation $\langle e_i, e\rangle_{1\le i\le n}$*, where* e_i *and* e *are boolean expressions, is a shorthand for* $\langle e_1, ..., e_n, e\rangle$*. We define the series* $U_k \in \mathsf{Bool}^{n+1}$ *associated to* B*,* B'*, and* δ *as follows:*

$$U_0 = \mathsf{true}^{n+1}, \qquad U_{k+1} = \langle (\nu\Phi_{\delta\oslash[(\Phi'_\delta(U_k))_{n+1}/X_{n+1}]})_i, (\Phi'_\delta(U_k))_{n+1}\rangle_{1\le i\le n}.$$

We are now ready to show the correctness of the BES encodings for weak τ-confluence. We start by proving the correctness of B_{wtc1}.

From Tarski's theorem [25], the maximal fixed point $\nu\Phi_{wtc1}$ can be computed as follows:

$$\nu\Phi_{wtc1} = \bigsqcup \{\boldsymbol{b} \in \mathsf{Bool}^{|\mathcal{S}|^2} \mid \boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})\}.$$

The following lemma provides a link between sets of confluent transitions and the functional associated to the Bes B_{wtc1}.

Lemma 1 *Let* $(\mathcal{S}, \mathcal{A}, \mathcal{T}, s_0)$ *be an* Lts*, and let* $\boldsymbol{b} \in \mathsf{Bool}^{|\mathcal{S}|^2}$*. Then:*

$$\boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b}) \;\; \textit{iff} \;\; \Gamma(\boldsymbol{b}) \textit{ is weak } \tau\textit{-confluent.}$$

Proof If. Let $\boldsymbol{b} = \langle b_{s_1,s_2}\rangle_{s_1,s_2\in\mathcal{S}}$ such that $\Gamma(\boldsymbol{b})$ is weak τ-confluent. We must show that $\boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})$.

Let $s_1, s_2 \in \mathcal{S}$ such that $b_{s_1,s_2} = \mathsf{true}$. From the definition of Γ, this implies $\langle s_1, s_2\rangle \in \Gamma(\boldsymbol{b})$. Since $\Gamma(\boldsymbol{b})$ is weak τ-confluent, from Definition 2 this implies that for all $s_1 \xrightarrow{a} s_3$, we have $s_1 \xrightarrow{\tau} s_2$, and for some $n, m, p > 0$ there exist $s_{2,0}, \ldots, s_{2,n}, s'_{2,0}, \ldots, s'_{2,m}, s_4 \in \mathcal{S}$ with $s_{2,0} = s_2$, and $s_{3,0}, \ldots, s_{3,p} \in \mathcal{S}$ with $s_{3,0} = s_3$ such that $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ and $\langle s_{2,i}, s_{2,i+1}\rangle \in S$ for $0 \le i < n$, $s_{2,n} \xrightarrow{\overline{a}} s'_{2,0}$, $s'_{2,j} \xrightarrow{\tau} s'_{2,j+1}$ and $\langle s'_{2,j}, s'_{2,j+1}\rangle \in S$ for $0 \le j < m$, $s'_{2,m} \xrightarrow{\tau} s_4$ and $\langle s'_{2,m}, s_4\rangle \in S$, $s_{3,k} \xrightarrow{\tau} s_{3,k+1}$ and $\langle s_{3,k}, s_{3,k+1}\rangle \in S$ for $0 \le k < p$, and $s_{3,p} \xrightarrow{\tau} s_4$ and $\langle s_{3,p}, s_4\rangle \in S$.

Let $s_1 \xrightarrow{a} s_3$ be a transition. From the condition above, for some $n, m, p > 0$ there exist $s_{2,0}, \ldots, s_{2,n}, s'_{2,0}, \ldots, s'_{2,m}, s_4 \in \mathcal{S}$ with $s_{2,0} = s_2$, and $s_{3,0}, \ldots, s_{3,p} \in \mathcal{S}$ with $s_{3,0} = s_3$ such that $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ for $0 \le i < n$, $s_{2,n} \xrightarrow{\overline{a}} s'_{2,0}$, $s'_{2,j} \xrightarrow{\tau} s'_{2,j+1}$ for $0 \le j < m$, $s'_{2,m} \xrightarrow{\tau} s_4$, $s_{3,k} \xrightarrow{\tau} s_{3,k+1}$ for $0 \le k < p$, and $s_{3,p} \xrightarrow{\tau} s_4$.

This means that $\langle s_{2,i}, s_{2,i+1}\rangle \in \Gamma(\boldsymbol{b})$, which from the definition of Γ implies $b_{s_{2,i},s_{2,i+1}} = \mathsf{true}$, for $0 \le i < n$, $\langle s'_{2,j}, s'_{2,j+1}\rangle \in \Gamma(\boldsymbol{b})$, which implies $b_{s'_{2,j},s'_{2,j+1}} = \mathsf{true}$, for $0 \le j < m$, $\langle s'_{2,m}, s_4\rangle \in \Gamma(\boldsymbol{b})$, which implies $b_{s'_{2,m},s_4} = \mathsf{true}$, $\langle s_{3,k}, s_{3,k+1}\rangle \in \Gamma(\boldsymbol{b})$, which implies $b_{s_{3,k},s_{3,k+1}} = \mathsf{true}$, for $0 \le k < p$, and $\langle s_{3,p}, s_4\rangle \in \Gamma(\boldsymbol{b})$, which implies $b_{s_{3,p},s_4} = \mathsf{true}$. Furthermore, it means that $\langle s_{3,k}, s_{3,k+1}\rangle \in \Gamma(\boldsymbol{b})$, which from the definition of Γ implies $b_{s_{3,k},s_{3,k+1}} = \mathsf{true}$, for $0 \le k < p$, and $\langle s_{3,p}, s_4\rangle \in \Gamma(\boldsymbol{b})$, which implies $b_{s_{3,p},s_4} = \mathsf{true}$. By definition of b_{s_1,s_2}, this means that the boolean formula $\bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2 \xrightarrow{\overline{a}} s''_2 \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}$ is also true. From the interpretation of boolean formulas given in Definition 3, this implies:

$$\langle [\![\bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2 \xrightarrow{\overline{a}} s''_2 \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}]\!] [b_{s_1,s_2}/X_{s_1,s_2}]\rangle_{s_1,s_2\in\mathcal{S}} = \mathsf{true}$$

meaning that $(\Phi_{wtc1}(\boldsymbol{b})) = \mathsf{true}$. Therefore, $\boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})$.

Only if. Let $\boldsymbol{b} = \langle b_{s_1,s_2}\rangle_{s_1,s_2\in\mathcal{S}}$ such that $\boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})$. We must show that $\Gamma(\boldsymbol{b})$ is weak τ-confluent, i.e., it satisfies Definition 2.

Let $\langle s_1, s_2\rangle \in \Gamma(\boldsymbol{b})$. From the definition of Γ, this implies $b_{s_1,s_2} = \mathsf{true}$. Since $\boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})$, from Definition 3 and the interpretation of boolean formulas, this implies:

$$\langle [\![\bigwedge_{s_1 \xrightarrow{a} s_3} \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2 \xrightarrow{\overline{a}} s''_2 \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}]\!] [b_{s_1,s_2}/X_{s_1,s_2}]\rangle_{s_1,s_2\in\mathcal{S}} = \mathsf{true}.$$

Let $s_1 \stackrel{a}{\rightarrow} s_3$ be a transition. From the condition above, each conjunct associated to such a transition must be true, i.e., $\bigvee_{s_2 \stackrel{\tau_c^*}{\rightarrow} s_2' \stackrel{\overline{a}}{\rightarrow} s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true} = \mathsf{true}$. This means that some disjunct corresponding to transitions $s_{2,i} \stackrel{\tau_c}{\rightarrow} s_{2,i+1}$ for some $n > 0$ with $0 \leq i < n$, $s_{2,0} = s_2$ and $s_{2,n} = s_2'$, $s_{2,n} \stackrel{\overline{a}}{\rightarrow} s_{2,0}'$, $s_{2,j}' \stackrel{\tau_c}{\rightarrow} s_{2,j+1}'$ for some $m > 0$ with $0 \leq j < m$ and $s_{2,0}' = s_2''$, $s_{2,m}' \stackrel{\tau_c}{\rightarrow} s_4$, $s_{3,k} \stackrel{\tau_c}{\rightarrow} s_{3,k+1}$ for some $p > 0$, with $0 \leq k < p$ and $s_{3,0} = s_3$, and $s_{3,p} \stackrel{\tau_c}{\rightarrow} s_4$, must be true. From the definition of b_{s_1,s_2}, this implies that for $0 \leq i < n$, $0 \leq j < m$, and $0 \leq k < p$, we have $b_{s_{2,i},s_{2,i+1}} = \mathsf{true}$, $b_{s_{2,j}',s_{2,j+1}'} = \mathsf{true}$, and $b_{s_{3,k},s_{3,k+1}} = \mathsf{true}$. Besides that, $b_{s_{2,m}',s_4} = \mathsf{true}$ and $b_{s_{3,p},s_4} = \mathsf{true}$. This, by the definition of Γ, implies that $\langle s_{2,i}, s_{2,i+1}\rangle \in \Gamma(\boldsymbol{b})$ for $0 \leq i < n$, $\langle s_{2,j}', s_{2,j+1}'\rangle \in \Gamma(\boldsymbol{b})$ for $0 \leq j < m$, $\langle s_{3,k}, s_{3,k+1}\rangle \in \Gamma(\boldsymbol{b})$ for $0 \leq k < p$, $\langle s_{2,m}', s_4\rangle \in \Gamma(\boldsymbol{b})$, and $\langle s_{3,p}, s_4\rangle \in \Gamma(\boldsymbol{b})$, which means that $\Gamma(\boldsymbol{b})$ satisfies Definition 2, and therefore $\Gamma(\boldsymbol{b})$ is a weak τ-confluent set. □

A useful property of weak τ-confluent sets is that they are closed under union, i.e., the union of two weak τ-confluent sets is also weak τ-confluent. This property can be easily shown for our notion of weak τ-confluence in the same way it was shown for τ-confluence in [11]. The proposition below states the correctness of the direct Bes encoding of weak τ-confluence.

Proposition 1 (Correctness of directly encoded weak τ-confluence BES) *Let $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{T}, s_0)$ be an* Lts*, and let B_{wtc1} be the* Bes *directly encoding a weak τ-confluent set of $\mathcal{M}$. Then:*

$$\Gamma([\![B_{wtc1}]\!]) = \mathbb{T}(\mathcal{S}).$$

Proof

$$\begin{array}{rlr}
\Gamma([\![B_{wtc1}]\!]) & = \Gamma(\nu\Phi_{wtc1}) & \text{by Definition 3} \\
& = \Gamma(\bigsqcup \{\boldsymbol{b} \mid \boldsymbol{b} \in \mathsf{Bool}^{|\mathcal{S}|^2} \wedge \boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})\}) & \text{by Tarski's theorem} \\
& = \bigcup \{\Gamma(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathsf{Bool}^{|\mathcal{S}|^2} \wedge \boldsymbol{b} \sqsubseteq \Phi_{wtc1}(\boldsymbol{b})\} & \text{by } \Gamma \text{ isomorphism} \\
& = \bigcup \{\Gamma(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathsf{Bool}^{|\mathcal{S}|^2} \wedge \Gamma(\boldsymbol{b}) \text{ is confluent}\} & \text{by Lemma 1} \\
& = \bigcup \{U \subseteq \mathcal{S}^2 \mid U \text{ is confluent}\} & \text{by } \Gamma \text{ bijection} \\
& = \mathbb{T}(\mathcal{S}) & \text{by closure under union.}
\end{array}$$

□

Before proving the correctness of B_{wtc}, we show first a lemma concerning the computation of τ-closures using boolean equations, and then we show the main proposition.

Lemma 2 *Let $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0\rangle$ be a τ-convergent* Lts*, and consider the* Bes*s:*

$$B = \left\{ \begin{array}{rcl}
Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \stackrel{\tau_c^*}{\rightarrow} s_2'} Z_{s_2',s_3,a} \\
U_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \stackrel{\tau_c^*}{\rightarrow} s_4} V_{s_3,s_4} \\
V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \stackrel{\tau_c^*}{\rightarrow} s_4} \mathsf{true}
\end{array} \right\}_{s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

and

$$B' = \left\{ \begin{array}{rcl} Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (X_{s_2,s'_2} \wedge Y_{s'_2,s_3,a}) \\ U_{s''_2,s_3} & \stackrel{\nu}{=} & V_{s_3,s''_2} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (X_{s''_2,s_4} \wedge U_{s_4,s_3}) \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (X_{s_3,s'_3} \wedge V_{s'_3,s_4}) \end{array} \right\}_{s_2,s''_2,s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Then:

$$[\![B]\!]\delta = [\![B']\!]\delta$$

for any context $\delta : \mathcal{X} \to \mathsf{Bool}$*, where the set* $\mathcal{X}$ *contains the variables* $Y_{s_2,s_3,a}$*,* $U_{s''_2,s_3}$*,* V_{s_3,s_4} *and* X_{s_1,s_2}*, and for all* $s_1, s_2 \in \mathcal{S}$*,* $\delta(X_{s_1,s_2}) = \mathsf{true}$ *iff* $s_1 \xrightarrow{\tau_c} s_2$*.*

Proof Let $\delta : \mathcal{X} \to \mathsf{Bool}$ be a context and $\Phi_\delta, \Phi'_\delta : \mathsf{Bool}^{|\mathcal{S}|^2 \cdot |\mathcal{A}|} \times \mathsf{Bool}^{|\mathcal{S}|^2} \times \mathsf{Bool}^{|\mathcal{S}|^2} \to \mathsf{Bool}^{|\mathcal{S}|^2 \cdot |\mathcal{A}|} \times \mathsf{Bool}^{|\mathcal{S}|^2} \times \mathsf{Bool}^{|\mathcal{S}|^2}$ be the functionals associated to B and B' in the context δ (for simplicity, we omit the subscript domains when their meaning is clear):

$$\begin{array}{rl} \Phi_\delta(\langle y_{s_2,s_3,a}, u_{s''_2,s_3}, v_{s_3,s_4} \rangle) = & \langle [\![\bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} Z_{s'_2,s_3,a}]\!] \\ & (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]), \\ & [\![\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4}]\!] \\ & (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]), \\ & [\![\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}]\!] (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]) \rangle \\ = & \langle \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}), \bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} v_{s_3,s_4}, \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \rangle \end{array}$$

$$\begin{array}{rl} \Phi'_\delta(\langle y_{s_2,s_3,a}, u_{s''_2,s_3}, v_{s_3,s_4} \rangle) = & \langle [\![Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (X_{s_2,s'_2} \wedge Y_{s'_2,s_3,a})]\!] \\ & \quad (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]), \\ & [\![V_{s_3,s''_2} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (X_{s''_2,s_4} \wedge U_{s_4,s_3})]\!] \\ & \quad (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]), \\ & [\![(s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (X_{s_3,s'_3} \wedge V_{s'_3,s_4})]\!] \\ & \quad (\delta \oslash [y_{s_2,s_3,a}/Y_{s_2,s_3,a}, u_{s''_2,s_3}/U_{s''_2,s_3}, v_{s_3,s_4}/V_{s_3,s_4}]) \rangle \\ = & \langle \delta(Z_{s_2,s_3,a}) \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (\delta(X_{s_2,s'_2}) \wedge y_{s'_2,s_3,a}), \\ & v_{s_3,s''_2} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (\delta(X_{s''_2,s_4}) \wedge u_{s_4,s_3}), \\ & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (\delta(X_{s_3,s'_3}) \wedge v_{s'_3,s_4}) \rangle \end{array}$$

To prove that $\nu\Phi_\delta = \nu\Phi'_\delta$, we show first that $\nu\Phi_\delta \sqsubseteq \nu\Phi'_\delta$ and then we show that the strict inclusion $\nu\Phi_\delta \sqsubset \nu\Phi'_\delta$ does not hold. It is clear that the functional Φ_δ of B is constant after two iterations; the equation for $Y_{s_2,s_3,a}$ is only dependent on δ, the equation for V_{s_3,s_4} contains no variables on the right-hand side, and the equation for $U_{s''_2,s_3}$ only depends on V_{s_3,s_4}, which is constant after the first iteration. Because of this, the maximal fixed point of Φ_δ is obtained simply by evaluating the functional on some arbitrary arguments:

$$\nu\Phi_\delta = \langle (\bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}))_{s_2,s_3,a}, (\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} ((\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4}))_{s''_2,s_3}, (\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4} \rangle$$

By applying Φ'_δ on this fixed point, we obtain:

$$\begin{aligned}
\Phi'_\delta(\nu\Phi_\delta) &= \Phi'_\delta(\langle \bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}), \bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} ((\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4}), \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\rangle) \\
&\hfill \text{by definition of } \nu\Phi_\delta \\
&= \langle \delta(Z_{s_2,s_3,a}) \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (\delta(X_{s_2,s'_2}) \wedge (\bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}))_{s'_2,s_3,a}), \\
&\quad (\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s''_2} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (\delta(X_{s''_2,s_4}) \wedge (\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} ((\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4}))_{s_4,s_3}), \\
&\quad (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (\delta(X_{s_3,s'_3}) \wedge (\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s'_3,s_4})\rangle \\
&\hfill \text{by definition of } \Phi'_\delta \\
&= \langle \delta(Z_{s_2,s_3,a}) \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (\delta(X_{s_2,s'_2}) \wedge \bigvee_{s'_2 \xrightarrow{\tau_c^*} s''_2} \delta(Z_{s''_2,s_3,a})), \\
&\quad \bigvee_{s_3 \xrightarrow{\tau_c^*} s''_2} \mathsf{true} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (\delta(X_{s''_2,s_4}) \wedge \bigvee_{s_4 \xrightarrow{\tau_c^*} s'_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s'_4} \mathsf{true}), \\
&\quad (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (\delta(X_{s_3,s'_3}) \wedge \bigvee_{s'_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})\rangle \\
&\hfill \text{by subscript substitution} \\
&= \langle \delta(Z_{s_2,s_3,a}) \vee \bigvee_{s_2 \xrightarrow{\tau_c} s'_2} \bigvee_{s'_2 \xrightarrow{\tau_c^*} s''_2} \delta(Z_{s''_2,s_3,a}), \\
&\quad \bigvee_{s_3 \xrightarrow{\tau_c^*} s''_2} \mathsf{true} \vee \bigvee_{s''_2 \xrightarrow{\tau_c} s_4} \bigvee_{s_4 \xrightarrow{\tau_c^*} s'_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s'_4} \mathsf{true}, \\
&\quad (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau_c} s'_3} \bigvee_{s'_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true}\rangle \\
&\hfill \text{by definition of } \tau_c \text{ and the assumption about } \delta \text{ concerning } X_{s_1,s_2} \\
&= \langle (\bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}))_{s_2,s_3,a}, (\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} ((\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4}))_{s''_2,s_3}, (\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true})_{s_3,s_4}\rangle \\
&\hfill \text{by definition of } \tau\text{-closure} \\
&= \nu\Phi_\delta \qquad \text{by definition of } \nu\Phi_\delta.
\end{aligned}$$

From Tarski's theorem [25], this implies $\nu\Phi_\delta \sqsubseteq \nu\Phi'_\delta$. It remains to show that the strict inclusion $\nu\Phi_\delta \sqsubset \nu\Phi'_\delta$ does not hold. Suppose that $\nu\Phi_\delta \sqsubset \nu\Phi'_\delta$, meaning that:

$$\langle (\nu\Phi_\delta)_{s_2,s_3,a}, (\nu\Phi_\delta)_{s''_2,s_3}, (\nu\Phi_\delta)_{s_3,s_4}\rangle \sqsubset \langle (\nu\Phi'_\delta)_{s_2,s_3,a}, (\nu\Phi'_\delta)_{s''_2,s_3}, (\nu\Phi'_\delta)_{s_3,s_4}\rangle$$

Three cases are possible, depending on whether the first, the second, or the third component of $\nu\Phi_\delta$ is smaller than the corresponding component of $\nu\Phi'_\delta$. We begin by considering the first case. Let $s_2, s_3 \in \mathcal{S}$ and $a \in \mathcal{A}$ such that $(\nu\Phi_\delta)_{s_2,s_3,a} = \mathsf{false}$ and $(\nu\Phi'_\delta)_{s_2,s_3,a} = \mathsf{true}$.

From the definition of Φ_δ, we infer that $\bigvee_{s_2 \xrightarrow{\tau_c^*} s'_2} \delta(Z_{s'_2,s_3,a}) = \mathsf{false}$, meaning that there is no τ-sequence going out of s_2 and leading to a state s'_2 such that $\delta(Z_{s'_2,s_3,a}) = \mathsf{true}$.

From the definition of Φ'_δ and the fact that $\nu\Phi'_\delta$ is a fixed point, we infer that $\delta(Z_{s_2,s_3,a}) \vee \bigvee_{s_2 \xrightarrow{\tau} s'_2} (\delta(X_{s_2,s'_2}) \wedge (\nu\Phi'_\delta)_{s'_2,s_3,a}) = \mathsf{true}$. But the disjunct $\delta(Z_{s_2,s_3,a})$ cannot be true because this would imply the existence of a zero-step τ-sequence going out of s_2 such that $\delta(Z_{s_2,s_3,a}) = \mathsf{true}$, which is forbidden by the condition above. So the other disjunct must be true, meaning that there exists a transition $s_2 \xrightarrow{\tau} s'_2$ such that $(\delta(X_{s_2,s'_2}) \wedge (\nu\Phi'_\delta)_{s'_2,s_3,a}) = \mathsf{true}$. By assumption, we know that $\delta(X_{s_2,s'_2}) = \mathsf{true}$, hence it remains to be shown that $(\nu\Phi'_\delta)_{s'_2,s_3,a} = \mathsf{true}$.

By repeating the above reasoning, we can construct an infinite sequence $(s =)s_{2,0} \xrightarrow{\tau} s_{2,1} \xrightarrow{\tau} s_{2,2} \xrightarrow{\tau} \cdots$ with $s_{2,0} = s_2$ and $s_{2,1} = s'_2$ such that $(\nu\Phi'_\delta)_{s_{2,i},s_3,a} = \mathsf{true}$ for all $i \geq 0$. This contradicts the hypothesis of $\mathcal{M}$ being τ-convergent, and therefore concludes the proof for the first case.

Next, we look at the third case. Let $s_3, s_4 \in \mathcal{S}$ and $a \in \mathcal{A}$ such that $(\nu\Phi_\delta)_{s_3,s_4} = \mathsf{false}$ and $(\nu\Phi'_\delta)_{s_3,s_4} = \mathsf{true}$.

From the definition of Φ_δ, we infer that $\bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} = \mathsf{false}$, meaning that there is no τ-sequence going out of s_3, leading to state s_4.

From the definition of Φ'_δ and the fact that $\nu\Phi'_\delta$ is a fixed point, we infer that $(s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (\delta(X_{s_3,s'_3}) \wedge (\nu\Phi'_\delta)_{s_3,s_4}) = \mathsf{true}$. But the disjunct $(s_3 = s_4)$ cannot be true because this would imply the existence of a zero-step τ-sequence going from s_3 to s_4, which is forbidden by the condition above. So the other disjunct must be true, meaning that there exists a transition $s_3 \xrightarrow{\tau} s'_3$ such that $(\delta(X_{s_3,s'_3}) \wedge (\nu\Phi'_\delta)_{s_3,s_4}) = \mathsf{true}$. By assumption, we know that $\delta(X_{s_3,s'_3}) = \mathsf{true}$, hence it remains to be shown that $(\nu\Phi'_\delta)_{s_3,s_4} = \mathsf{true}$.

By repeating the above reasoning, we can construct an infinite sequence $(s =)s_{3,0} \xrightarrow{\tau} s_{3,1} \xrightarrow{\tau} s_{3,2} \xrightarrow{\tau} \cdots$ with $s_{3,0} = s_3$ and $s_{3,1} = s'_3$ such that $(\nu\Phi'_\delta)_{s_{3,i},s_4} = \mathsf{true}$ for all $i \geq 0$. This contradicts the hypothesis of $\mathcal{M}$ being τ-convergent, and therefore concludes the proof for the third case.

For the second case, let $s''_2, s_3 \in \mathcal{S}$ and $a \in \mathcal{A}$ such that $(\nu\Phi_\delta)_{s''_2,s_3} = \mathsf{false}$ and $(\nu\Phi'_\delta)_{s''_2,s_3} = \mathsf{true}$.

From the definition of Φ_δ and the fact that $\nu\Phi_\delta$ is a fixed point, we infer that $\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} (\nu\Phi_\delta)_{s_3,s_4} = \mathsf{false}$, i.e. $\bigvee_{s''_2 \xrightarrow{\tau_c^*} s_4} \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} = \mathsf{false}$, meaning that there is no τ-sequence going out of s''_2, leading to a state s_4 such that there exists a τ-sequence from s_3 to s_4.

From the definition of Φ'_δ and the fact that $\nu\Phi'_\delta$ is a fixed point, we infer that $\delta(\nu\Phi'_\delta)_{s_3,s''_2} \vee \bigvee_{s''_2 \xrightarrow{\tau} s_4} (\delta(X_{s''_2,s_4}) \wedge (\nu\Phi'_\delta)_{s_4,s_3}) = \mathsf{true}$. Based on the definition of Φ'_δ, we distinguish two possibilities:

1. $\delta(\nu\Phi'_\delta)_{s_3,s''_2} = \mathsf{true}$. In this possibility, by definition of the third component of Φ'_δ, and the fact that $\nu\Phi'_\delta$ is a fixed point, we infer that $(s_3 = s''_2) \vee \bigvee_{s_3 \xrightarrow{\tau} s'_3} (\delta(X_{s_3,s'_3}) \wedge (\nu\Phi'_\delta)_{s'_3,s''_2}) = \mathsf{true}$. But the disjunct $(s_3 = s''_2)$ cannot be true because this would imply the existence of zero-step τ-sequences going from s''_2 and s_3 to s_4, which is forbidden by the condition above. So the other disjunct must be true, meaning that there exists a transition $s_3 \xrightarrow{\tau} s'_3$ such that $(\delta(X_{s_3,s'_3}) \wedge (\nu\Phi'_\delta)_{s'_3,s''_2}) = \mathsf{true}$. By assumption, we know that $\delta(X_{s_3,s'_3}) = \mathsf{true}$, hence it remains to be shown that $(\nu\Phi'_\delta)_{s'_3,s''_2} = \mathsf{true}$.

   By repeating the above reasoning, we can construct an infinite sequence $(s =)s_{3,0} \xrightarrow{\tau} s_{3,1} \xrightarrow{\tau} s_{3,2} \xrightarrow{\tau} \cdots$ with $s_{3,0} = s_3$ and $s_{3,1} = s'_3$ such that $(\nu\Phi'_\delta)_{s_{3,i},s''_2} = \mathsf{true}$ for all $i \geq 0$. This contradicts the hypothesis of $\mathcal{M}$ being τ-convergent, and therefore concludes the proof for this possibility of the third case.

2. $\delta(\nu\Phi'_\delta)_{s_3,s''_2} = \mathsf{false}$. In this possibility, $\bigvee_{s''_2 \xrightarrow{\tau} s_4} (\delta(X_{s''_2,s_4}) \wedge (\nu\Phi'_\delta)_{s_4,s_3}) = \mathsf{true}$ should hold, meaning that there exists a transition $s''_2 \xrightarrow{\tau} s_4$ such that $(\delta(X_{s''_2,s_4}) \wedge (\nu\Phi'_\delta)_{s_4,s_3}) =$

true. By assumption, we know that $\delta(X_{s_2'',s_4}) = \mathsf{true}$, hence it remains to be shown that $(\nu\Phi'_\delta)_{s_4,s_3} = \mathsf{true}$.

By repeating the above reasoning, each time, we can distinguish two possibilities; since the first possibility always infers an infinite τ-sequence from s_3, contradicting the hypothesis of $\mathcal{M}$ being τ-convergent, we have to consider the second possibility every time. By this, we can construct an infinite sequence $(s =)s_{2,0} \xrightarrow{\tau} s_{2,1} \xrightarrow{\tau} s_{2,2} \xrightarrow{\tau} \cdots$ with $s_{2,0} = s_2''$ and $s_{2,1} = s_4$ such that $(\nu\Phi'_\delta)_{s_{2,i},s_3} = \mathsf{true}$ for all $i \geq 0$. This contradicts the hypothesis of $\mathcal{M}$ being τ-convergent, and therefore concludes the proof for this possibility of the third case.

□

Proposition 2 (Correctness of weak τ-confluence BES) *Let $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, s_0 \rangle$ be a τ-convergent* LTS*, and let B_{wtc} be the* BES *encoding the weak τ-confluent set of $\mathcal{M}$. Then:*

$$\Gamma([\![B_{wtc}]\!]) = \mathbb{T}(\mathcal{S}).$$

Proof By Proposition 1, we have $[\![B_{wtc1}]\!] = \mathbb{T}(\mathcal{S})$. Note that this proof is valid for arbitrary LTSs, and therefore also for τ-convergent LTSs. From this, we progressively refine B_{wtc1} until obtaining the full BES encoding given by Definition 4.

We now refine the BES B_{wtc1} into a BES B_{wtc2} by replacing certain subformulas with new variables defined by additional equations, such that the right-hand side of each equation of B_{wtc2} contains a single type of boolean operator:

$$B_{wtc2} = \left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2' \xrightarrow{\overline{a}} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

This transformation into a simple BES B_{wtc2} does not change the interpretation of the variables X_{s_1,s_2} of the original BES, i.e., $([\![B_{wtc1}]\!])X_{s_1,s_2} = ([\![B_{wtc2}]\!])X_{s_1,s_2}$ for all $s_1, s_2 \in \mathcal{S}$.

The final step towards the BES B_{wtc} given in Definition 4 is to get rid of the τ-closures present in the right-hand sides of the equations defining $Y_{s_2,s_3,a}$, U_{s_2',s_3}, and V_{s_3,s_4}. To achieve this, we consider the following BES:

$$B = \left\{ \begin{array}{rcl} Y_{s_2,s_3,a} & \stackrel{\nu}{=} & \bigvee_{s_2 \xrightarrow{\tau_c^*} s_2'} Z_{s_2',s_3,a} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & \bigvee_{s_2'' \xrightarrow{\tau_c^*} s_4} V_{s_3,s_4} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & \bigvee_{s_3 \xrightarrow{\tau_c^*} s_4} \mathsf{true} \end{array} \right\}_{s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Since LTS $\mathcal{M}$ is τ-convergent, Lemma 2 ensures that $[\![B]\!]\delta = [\![B']\!]\delta$, where B' is defined as follows:

$$B' = \left\{ \begin{array}{rcl} Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} (X_{s_2,s_2'} \wedge Y_{s_2',s_3,a}) \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} (X_{s_2'',s_4} \wedge U_{s_4,s_3}) \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} (X_{s_3,s_3'} \wedge V_{s_3',s_4}) \end{array} \right\}_{s_2,s_2'',s_3,s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

Starting with the BESs B and B', which have the same interpretation in any context δ, we can apply Lemma 4 of the full version of [19] repeatedly in order to add all the equations of B_{wtc2} defining variables X_{s_1,s_2} and $Z_{s_2',s_3,a}$ for all $s_1, s_2, s_2', s_3 \in \mathcal{S}$ and $a \in \mathcal{A}$, still ensuring that the resulting BESs have the same interpretation. Furthermore, we can add additional equations $Y_{s_2,s_2',s_3,a}$, $U_{s_2',s_4,a}$, and V_{s_3,s_3',s_4} for all $s_2, s_2', s_3, s_3', s_4 \in \mathcal{S}$ and $a \in \mathcal{A}$ in order to remove the conjunctions from $Y_{s_2,s_3,a}$, U_{s_2',s_3}, and V_{s_3,s_4} in B', respectively. Upon completion of this process, the BES derived from B is B_{wtc2} and the BES derived from B' is B_{wtc} (note that we can permute freely the equations of a BES without changing the interpretation of its variables):

$$B_{wtc} = \left\{ \begin{array}{rcl} X_{s_1,s_2} & \stackrel{\nu}{=} & \bigwedge_{s_1 \xrightarrow{a} s_3} Y_{s_2,s_3,a} \\ Y_{s_2,s_3,a} & \stackrel{\nu}{=} & Z_{s_2,s_3,a} \vee \bigvee_{s_2 \xrightarrow{\tau} s_2'} Y'_{s_2,s_2',s_3,a} \\ Y'_{s_2,s_2',s_3,a} & \stackrel{\nu}{=} & X_{s_2,s_2'} \wedge Y_{s_2',s_3,a} \\ Z_{s_2',s_3,a} & \stackrel{\nu}{=} & (a = \tau \wedge U_{s_2',s_3}) \vee \bigvee_{s_2' \xrightarrow{a} s_2''} U_{s_2'',s_3} \\ U_{s_2'',s_3} & \stackrel{\nu}{=} & V_{s_3,s_2''} \vee \bigvee_{s_2'' \xrightarrow{\tau} s_4} U'_{s_2'',s_4,s_3} \\ U'_{s_2'',s_4,s_3} & \stackrel{\nu}{=} & X_{s_2'',s_4} \wedge U_{s_4,s_3} \\ V_{s_3,s_4} & \stackrel{\nu}{=} & (s_3 = s_4) \vee \bigvee_{s_3 \xrightarrow{\tau} s_3'} V'_{s_3,s_3',s_4} \\ V'_{s_3,s_3',s_4} & \stackrel{\nu}{=} & X_{s_3,s_3'} \wedge V_{s_3',s_4} \end{array} \right\}_{s_1,s_2,s_2',s_2'',s_3,s_3',s_4 \in \mathcal{S}, a \in \mathcal{A}}$$

By virtue of Lemma 4 of the full version of [19], these BESs have the same interpretation (the context δ becomes useless since both BESs are closed), meaning that B_{wtc} has in turn the same interpretation as B_{wtc1}, and thus it reflects the weak τ-confluent set of the τ-convergent LTS $\mathcal{M}$ correctly. □

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